Sec 13 رياضيات

## Gamma Function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

1. $\Gamma(1) = 1$ , $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
2. $\Gamma(n) = (n-1)!$
3. $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$
4. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

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(1) $\alpha$ عدد سالب $\qquad \Gamma(\alpha) = \dfrac{\Gamma(\alpha+1)}{\alpha}$

(2) $n$ عدد صحيح موجب $\qquad \Gamma(n) = (n-1)!$

(3) $\alpha$ عدد كسر موجب $\qquad \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

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* $\Gamma(9) = 8!$
* $\Gamma(\frac{5}{2}) = \frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$
* $\Gamma(\frac{-5}{2}) = \frac{-2}{5}\cdot\frac{-2}{3}\cdot\frac{-1}{2}\cdot\Gamma(\frac{1}{2})$
* $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \dfrac{\pi}{\sin(\frac{\pi}{3})}$

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الأفكار

(1) $e^{جسم(x)}$ نعوض عن جسم$(x)$ ب $-u$

(2) $a^{F(x)} = e^{\ln a^{F(x)}}$

(3) $\ln(x) \leftarrow \ln(x) = -t$

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1. $\int_0^\infty x^3 e^{-x}\, dx = \Gamma(4) = 3!$

2. $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$

let $x^3 = u \rightarrow x = u^{\frac{1}{3}}$

$dx = \frac{1}{3} u^{\frac{-2}{3}} du$

$\frac{1}{3}\int_0^\infty u^{\frac{1}{6}} \cdot u^{\frac{-2}{3}} e^{-u}\, du$

$= \frac{1}{3}\int_0^\infty u^{\frac{-3}{6}} e^{-u}\, du = \frac{1}{3}\Gamma(\frac{-3}{6})$

$= \frac{1}{3}\Gamma(\frac{-1}{2}) = \frac{1}{3}\cdot\frac{-2}{1}\cdot\sqrt{\pi}$

3. $\int_0^{\frac{1}{2}} x^{m-1} \ln(\frac{1}{2x})\, dx$

$= -\int_0^{\frac{1}{2}} x^{m-1} \ln(2x)\, dx$

let $\ln(2x) = -t$

$x = \frac{1}{2} e^{-t}$

$dx = \frac{-1}{2} e^{-t} dt$

at $x = 0 \rightarrow t = \infty$

at $x = \frac{1}{2} \rightarrow t = 0$

$= -\int_\infty^0 (\frac{1}{2})^{m-1} e^{-(m-1)t} \cdot (-t)(\frac{-1}{2}) e^{-t}\, dt$

$= +(\frac{1}{2})^m \int_0^\infty t\, e^{-mt}\, dt$

Let $u = mt$

$dt = \frac{1}{m} du$

$= (\frac{1}{2})^m \int_0^\infty \frac{1}{m} u \cdot e^{-u} \cdot \frac{1}{m}\, du$

$= (\frac{1}{2})^m \cdot \frac{1}{m^2}\Gamma(2)$

$= (\frac{1}{2})^m \cdot \frac{1}{m^2}$

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## Beta Function

1. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$
2. $\beta(m,n) = 2\int_0^{\pi} \sin^{2m-1}\theta \cos^{2n-1}\theta\, dx$
3. $\beta(m,n) = \int_0^\infty \dfrac{y^{m-1}}{(1-y)^{m+n}}\, dy$
4. $\beta(m,n) = \beta(n,m)$
5. $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Ex $\int_0^1 x^5 (1-x)^3 \, dx$

$$= \beta(6,4) = \frac{\Gamma(6)\Gamma(4)}{\Gamma(10)}$$

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$$\int_0^1 \frac{5x}{\sqrt{1-x^5}} \, dx$$

Let $x^5 = t \rightarrow x = t^{\frac{1}{5}}$

$$dx = \frac{1}{5} t^{-\frac{4}{5}}$$

$$\frac{1}{5}\int_0^1 \frac{t^{-\frac{4}{5}+\frac{1}{5}}}{\sqrt{1-t}} \, dt = \frac{1}{5}\int_0^1 t^{-\frac{3}{5}} (1-t)^{-\frac{1}{2}} \, dt$$

$$= \beta\left(\frac{2}{5}, \frac{1}{2}\right) = \frac{\Gamma(\frac{2}{5})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{9}{10})}$$

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$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{\sin^3\theta}{\cos\theta}} \, d\theta = \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}}\theta \, \cos^{-\frac{1}{2}}\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{5}{4}, \frac{1}{4}\right) = \frac{\Gamma(\frac{5}{4})\,\Gamma(\frac{1}{4})}{\Gamma(\frac{6}{4})}$$

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## Bessel function

$$x^2 y'' + x y' + (x^2 - k) y = 0$$

$$J_k(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+k}}{r! \, (r+k+1)}$$

→ $k_1 - k_2$ = non Integer

$$y_{G.S} = c_1 J_n(x) + c_2 J_{-k}(x)$$

→ $k_1 - k_2$ = Integer

$$y_{G.S} = c_1 J_k(x) + c_2 y_n(x)$$

$$y_k(x) = \lim_{n \to k} \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}$$